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Capillary–gravity simultons in a liquid layer

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Simultaneous capillary–gravity solitary waves (simultons or quadratic solitons) are shown to be possible in a rectangular liquid channel of arbitrary finite depth bounded below by a solid plate and above with a free deformable surface with constant surface tension. A second-harmonic resonance between two waveguide modes (fundamental and second-harmonic waves) is studied with the inclusion of dispersion in the system. The nonlinearly coupled amplitude equations for the two slowly varying envelopes of the fundamental and the second-harmonic wave components are derived using the method of multiple scales. Two types of capillary–gravity simulton solutions are explicitly obtained and an experiment for observing such hydrodynamic simultons is suggested.

1. Introduction

In the past few decades, the nonlinear dynamics of water waves has been intensively studied, including modulational instabilities, transverse and dilational surface waves, various types of solitons, mode–mode resonant interactions, etc. (Akylas, Dias & Grimshaw 1998; Benjamin & Feir 1967; Bridges, Dias & Menasce 2001; Chossat & Dias 1995; Christodoulides & Dias 1994; Chu & Velarde 1988; Craik 1985; Davey & Stewartson 1974; Davis 1987; Debnath 1984; Dias & Kharif 1999; Hammack & Henderson 1993; Johnson 1997; Jones 1993, 1994; Lighthill 1978; Mei 1989; Miles 1980, 1981; Miles & Henderson 1990; Nepomnyashchy & Velarde 1994; Nepomnyashchy, Velarde & Colinet 2002; Oron, Davis & Bankoff 1997; Rednikov *et al.* 2000; Velarde, Nepomnyashchy & Hennenberg 2000; Whitham 1974; Wu 2000). An important resonant interaction is the second-harmonic generation (SHG), a degenerate case of general three-wave resonances. The SHG in inviscid deep water with surface tension was first investigated by Simmons (1969), McGoldrick (1970*a, b*) and Nayfeh (1970). McGoldrick (1970*b*) pointed out that in Wilton's approach to ripples (Wilton 1915) the appearance of the singularities in the Stokes expansion for large-amplitude capillary-gravity waves corresponded to an SHG and some higher-order resonances. The study of the SHG and other mode–mode interaction processes in water waves is of interest in understanding the spectrum of oceanic waves (Mei 1989).

For an inviscid fluid with a free surface, the Euler equations with the boundary conditions on the surface are of second-order nonlinearity. Thus an SHG is possible if an appropriate phase-matching condition is satisfied. Simmons (1969), McGoldrick (1970*a, b*) and Nayfeh (1970) have shown this possibility by considering an open liquid layer with surface tension and discussed in detail the interaction between two phase-matched *plane waves*. In recent years there has been increased interest in cascading phenomena related to second-harmonic interactions (Huang 2001*a,b*; Konotop & Malomed 2000). The physics of such a process requires two successive second-order processes in order for the net output to return to the input frequency (ω). This can occur via up-conversion ($\omega + \omega \rightarrow 2\omega$, i.e. SHG) followed by down-conversion ($2\omega - \omega \rightarrow \omega$) or via down-conversion ($\omega - \omega \rightarrow 0$, i.e. a rectification process) followed by up-conversion ($\omega + 0 \rightarrow \omega$). Such cascading processes may result in unexpected nonlinear excitations in the system. In this paper we show that two solitary wave modes appearing on the water surface are simultaneously possible in an SHG process they are called quadratic solitons or hydrodynamic simultons and define a new type of nonlinear capillary-gravity wave excitation in fluid layers with finite depth. The paper is organized as follows. In § 2 we present our model and make an asymptotic expansion based on the method of multiple scales. In § 3 we solve the nonlinearly coupled amplitude equations, which are derived in § 2 when considering the interaction of the fundamental and the second-harmonic waves with inclusion of dispersion in the system. Two types of hydrodynamic simulton solutions are explicitly provided. Then we discuss in § 4 a parametrically excited system and suggest an experiment for observing the predicted hydrodynamic simultons. Finally, in the last section a discussion and summary of results are presented.

2. Model and asymptotic expansion

2.1. *The model*

We consider the irrotational motion of an incompressible inviscid fluid layer in a gravitational field. The fluid at rest fills a horizontal rectangular channel to the depth d with $-d < z < 0$, where z is the vertical coordinate, b is the width along the transverse coordinate y, and the channel is of infinite extent along the other transverse coordinate, x. Shown in figure 1 is a schematic representation of the system. The surface of the liquid is open to ambient air and deformable with non-vanishing but constant surface tension, α. Air is considered hydrodynamically passive. The velocity potential ϕ of the fluid satisfies the Laplace equation

$$\nabla^2\phi = 0 \quad \text{for} \quad -d < z < \zeta(x, y, t), \tag{2.1}$$

with the boundary conditions on two lateral sides and bottom of the channel

$$\phi_y = 0 \quad \text{at} \quad y = 0, b, \tag{2.2}$$

$$\phi_z = 0 \quad \text{at} \quad z = -d, \tag{2.3}$$

and the kinematic and dynamic boundary conditions on the free surface $z = \zeta(x, y, t)$

$$\zeta_t + \phi_x\zeta_x + \phi_y\zeta_y = \phi_z, \tag{2.4}$$

$$\phi_t + g\zeta + \tfrac{1}{2}(\nabla\phi)^2 = \sigma\frac{\zeta_{xx}(1+\zeta_y^2) + \zeta_{yy}(1+\zeta_x^2) - 2\zeta_x\zeta_y\zeta_{xy}}{(1+\zeta_x^2+\zeta_y^2)^{3/2}}, \tag{2.5}$$

where the subscripts represent partial derivatives, e.g. $\phi_x = \partial\phi/\partial x$, etc; g is the acceleration due to gravity, and $\sigma = \alpha/\rho$ with α the surface tension and ρ the density of the fluid.

For an infinitesimal surface wave excitation one can easily solve (2.1)–(2.5). The solution takes the form given in (2.31) and (2.32) in § 2.3 below. The linear dispersion relation reads

$$\omega^2 = k(g + \sigma k^2)\tanh(kd), \tag{2.6}$$

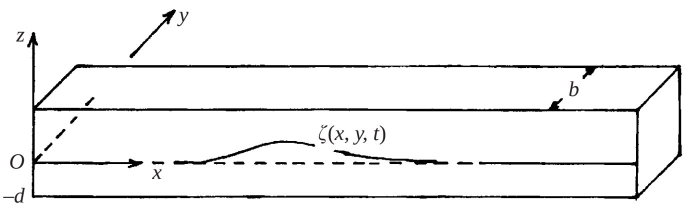

FIGURE 1. A schematic representation of the system under study.

where k and ω are the wavenumber and frequency, respectively; $k^2 = k_x^2 + k_y^2$, with k_x an arbitrary real number and $k_y = n\pi/b$, where n is an integer. From (2.6) we see that $\omega = \omega(k_x, k_y) = \omega(k_x, n\pi/b) \equiv \omega_n(k_x)$, i.e. many dispersion curves exist denoted by the branch index n. For the nth branch there exists a lower cutoff frequency $\omega_{nc} = \omega_n(0)$. Thus our system is a typical waveguide for water wave propagation. Non-propagating solitons related to the (0,1)-mode (i.e. $k_x = 0$ and $n = 1$) have been discovered in such a system (Wu *et al.* 1984) and many remarkable properties of these solitons have been investigated both in experiments (Denardo, Wright & Putterman 1990; Chen & Wei 1994, 1996; Wang & Wei 1997) and theoretically (Larraza & Putterman 1984; Miles 1984; Huang, Yan & Dai 1990; Miao & Wei 1999).

We look for possible hydrodynamic simulton excitations in the system. Accordingly, we consider two wave modes for which an SHG can occur. Necessary conditions for the SHG are the phase-matching conditions

$$\boldsymbol{k}_2 = 2\boldsymbol{k}_1, \tag{2.7}$$

$$\omega(\boldsymbol{k}_2) = 2\omega(\boldsymbol{k}_1), \tag{2.8}$$

where $\boldsymbol{k}_1$ (respectively, $\boldsymbol{k}_2$) is the wavevector corresponding to the fundamental (respectively, second-harmonic) wave. Besides (2.7) and (2.8), the following group-velocity matching is also necessary for exciting the hydrodynamic simultons:

$$\boldsymbol{v}_g(\boldsymbol{k}_2) = \boldsymbol{v}_g(\boldsymbol{k}_1), \tag{2.9}$$

where $\boldsymbol{v}_g(\boldsymbol{k}) = \mathrm{d}\omega/\mathrm{d}\boldsymbol{k}$ is the group velocity of the respective mode. Note that if the group-velocity difference between $\boldsymbol{v}_g(k_2)$ and $\boldsymbol{v}_g(k_1)$ is high, the fundamental and the second-harmonic waves will quickly separate from each other. Thus if (2.9) is not fulfilled an efficient energy exchange between the two wave modes mentioned above cannot be realized and hence a hydrodynamic simulton cannot be formed (Konotop & Malomed 2000). In our system, the modes satisfying condition (2.9) are the cutoff modes, i.e. $\boldsymbol{k} = (0, n\pi/b)$. These modes have in fact equal (zero) group velocity. To meet the conditions (2.7) and (2.8), we chose, for simplicity, $\boldsymbol{k}_1 = (0, k_1)$ and $\boldsymbol{k}_2 = (0, 2k_1)$ with $k_1 = \pi/b$. Then the conditions (2.7) and (2.8) require

$$f(k_1^*) \equiv \tanh^2 k_1^* - \frac{3(k_1^*/k_0^*)^2}{1 + (k_1^*/k_0^*)^2} = 0, \tag{2.10}$$

where $k_1^* = k_1 d$ and $k_0^* = d(g/\sigma)^{1/2}$, both of which (and hence $f(k_1^*)$) are dimensionless. Shown in figure 2 is the function $f(k_1^*)$ for different layer depths for water, $d =$ 2 cm, 2.5 cm, 3 cm, and 4 cm. Note that when the depth d changes the value of the dimensionless parameter k_0^* also changes, resulting in a different dimensionless function $f(k_1^*)$. We can see that the function $f(k_1^*)$ has two zero points. The first one is at $k_1^* = 0$, not relevant for the SHG. The other zero point is at a non-vanishing value of k_1^* and hence corresponds to the sought SHG. The condition (2.10)

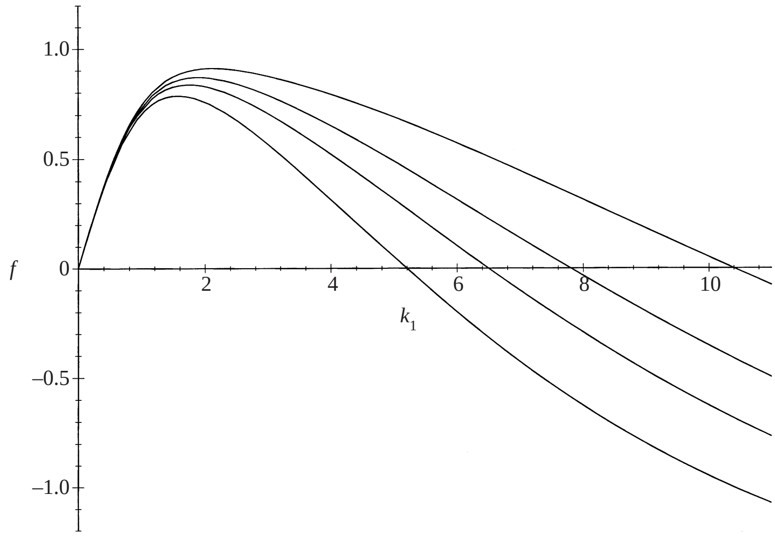

FIGURE 2. The function $f(k_1^*)$ for different depths of the liquid layer. From the lower to upper level the curves correspond respectively to $d = 2$ cm, 2.5 cm, 3 cm, and 4 cm for water with $\rho_{water} = 1\,\mathrm{g\,cm^{-3}}$ and $\alpha_{water} = 72.5\,\mathrm{dyn\,cm^{-1}}$.

imposes a constraint on the parameters of the system. For example for water at room temperature we have $\rho_{water} = 1\,\mathrm{g\,cm^{-3}}$ and $\alpha_{water} = 72.5\,\mathrm{dyn\,cm^{-1}}$. Taking $d = 2$ cm and $g = 980\,\mathrm{cm\,s^{-2}}$, we obtain $k_0^* = 7.35$. Then the dimensionless wavenumber corresponding to the SHG is $k_1^* = 5.20$. Thus the realization of the SHG requires that the width b $(= \pi d/k_1^*)$ equals 1.2083 cm. For $d = 2.5$ cm, 3.0 cm, and 4.0 cm we obtain almost the same value $b = 1.2084$ cm. In the limit of an infinitely deep water layer, (2.10) has the exact solution $k_1 = [g/(2\sigma)]^{1/2}$ and hence $\omega_1 = [9g^3/(8\sigma)]^{1/4}$, corresponding to $b = (2\sigma/g)^{1/2}\pi$, which has the value 1.20843 cm. Thus if one designs the channel with the width b near 1.21 cm and the initially static water depth $d \geqslant 2$ cm, simultons should be experimentally observable.

2.2. *Asymptotic expansion*

We apply the method of multiple scales to investigate the SHG for the system (2.1)–(2.5). Since we are interested in a cascading process in which the excitation width is smaller than that in the usual SHG case, we use different multiple-scale variables and different asymptotic expansions to characterize the evolution of the amplitudes of the fundamental and the second-harmonic waves. Appropriate multiple-scale variables and asymptotic expansion are

$$\xi = \epsilon^{1/2}(x - v_g t), \tag{2.11}$$

$$\tau = \epsilon t, \tag{2.12}$$

$$\phi = \epsilon(\phi^{(0)} + \epsilon^{1/2}\phi^{(1)} + \epsilon\phi^{(2)} + \cdots), \tag{2.13}$$

$$\zeta = \epsilon(\zeta^{(0)} + \epsilon^{1/2}\zeta^{(1)} + \epsilon\zeta^{(2)} + \cdots), \tag{2.14}$$

where v_g is a constant determined by a solvability condition, ϵ is the smallness parameter measuring the slope of the wavy surface, $\phi^{(j)}$ and $\zeta^{(j)}$ $(j = 0, 1, 2, \cdots)$ are functions of the fast variables x, y, z and t and the slow variables ξ and τ. Substituting

(2.11)–(2.14) into (2.1)–(2.3), we have

$$\nabla^2\phi^{(j)} = P^{(j)} \quad \text{for} \quad -d < z < \zeta, \tag{2.15}$$

$$\phi_y^{(j)} = 0 \quad \text{at} \quad y = 0,\ b, \tag{2.16}$$

$$\phi_z^{(j)} = 0 \quad \text{at} \quad z = -d, \tag{2.17}$$

with

$$P^{(0)} = 0, \tag{2.18}$$

$$P^{(1)} = -2\phi_{x\xi}^{(0)}, \tag{2.19}$$

$$P^{(2)} = -2\phi_{x\xi}^{(1)} - \phi_{\xi\xi}^{(0)}, \tag{2.20}$$

$$\cdots .$$

To expand the boundary conditions on the free surface we first take a Taylor expansion for ϕ at $z = 0$ and use (2.11)–(2.14). Then (2.4) and (2.5) become

$$\zeta_t^{(j)} - \phi_z^{(j)} = Q^{(j)}, \tag{2.21}$$

$$\phi_t^{(j)} + g\zeta^{(j)} - \sigma(\zeta_{xx}^{(j)} + \zeta_{yy}^{(j)}) = R^{(j)}, \tag{2.22}$$

at $z = 0$, with $j = 0, 1, 2, \cdots$ and

$$Q^{(0)} = 0, \tag{2.23}$$

$$Q^{(1)} = v_g\zeta_\xi^{(0)}, \tag{2.24}$$

$$Q^{(2)} = v_g\zeta_\xi^{(1)} - \zeta_\tau^{(0)} - \phi_x^{(0)}\zeta_x^{(0)} - \phi_y^{(0)}\zeta_y^{(0)} + \phi_{zz}^{(0)}\zeta^{(0)}, \tag{2.25}$$

$$\cdots ,$$

and

$$R^{(0)} = 0, \tag{2.26}$$

$$R^{(1)} = v_g\phi_\xi^{(0)} + 2\sigma\zeta_{x\xi}^{(0)}, \tag{2.27}$$

$$R^{(2)} = v_g\phi_\xi^{(1)} - \phi_\tau^{(0)} + \sigma(2\zeta_{x\xi}^{(1)} + \zeta_{\xi\xi}^{(0)}) - \phi_{tz}^{(0)}\zeta^{(0)} - \tfrac{1}{2}(\nabla\phi^{(0)})^2, \tag{2.28}$$

$$\cdots .$$

Then (2.21) and (2.22) can be rewritten in the following form:

$$\phi_{tt}^{(j)} + [g - \sigma(\partial_x^2 + \partial_y^2)]\phi_z^{(j)} = R_t^{(j)} - [g - \sigma(\partial_x^2 + \partial_y^2)]Q^{(j)}, \tag{2.29}$$

$$\zeta^{(j)} = \int \mathrm{d}t(Q^{(j)} + \phi_z^{(j)}), \tag{2.30}$$

at $z = 0$. From (2.15)–(2.17) follows $\phi^{(j)}$. For different j the boundary condition (2.29) gives a series of solvability conditions, including the linear dispersion relation and the amplitude equations. Using (2.30) one can obtain the surface displacement $\zeta^{(j)}$ through $\phi^{(j)}$.

2.3. *Amplitude equations for cascading processes*

We now solve (2.15)–(2.17), together with the boundary conditions (2.29) and (2.30). At leading order ($j=0$) the solution reads

$$\phi^{(0)} = \frac{\cosh k(z+d)}{\cosh kd} \cos k_y y \{A(\xi,\tau) \exp[\mathrm{i}(k_x x - \omega t)] + \text{c.c.}\}, \tag{2.31}$$

$$\zeta^{(0)} = \mathrm{i}(kT/\omega) \cos k_y y \{A(\xi,\tau) \exp[\mathrm{i}(k_x x - \omega t)] - \text{c.c.}\}, \tag{2.32}$$

$$\omega^2 = k(g+\sigma k^2)T, \quad T = \tanh kd, \tag{2.33}$$

where $k^2 = k_x^2 + k_y^2$, k_x is an arbitrary real number and $k_y = n\pi/b$ with n being an arbitrary integer; A is an amplitude function of the slow variables ξ and τ left undetermined; c.c. represents the corresponding complex conjugate term. For an SHG in the system we consider two modes, e.g. the (0,1) and (0,2) modes, which correspond to $k_x = 0$, $n = 1$ and $k_x = 0$, $n = 2$, respectively. Then the solution (2.31) and (2.32) is taken as

$$\phi^{(0)} = \frac{\cosh k_1(z+d)}{\cosh k_1 d} \cos k_1 y \{A_1(\xi,\tau) \exp[-\mathrm{i}\omega_1 t)] + \text{c.c.}\} + \frac{\cosh k_2(z+d)}{\cosh k_2 d} \cos k_2 y \{A_2(\xi,\tau) \exp[-\mathrm{i}\omega_2 t)] + \text{c.c.}\}, \tag{2.34}$$

$$\zeta^{(0)} = \mathrm{i}(k_1 T_1/\omega_1) \cos k_1 y \{A_1(\xi,\tau) \exp[-\mathrm{i}\omega_1 t)] - \text{c.c.}\} + \mathrm{i}(k_2 T_2/\omega_2) \cos k_2 y \{A_2(\xi,\tau) \exp[-\mathrm{i}\omega_2 t)] - \text{c.c.}\}, \tag{2.35}$$

where $k_1 = \pi/b$, $k_2 = 2k_1 = 2\pi/b$, $\omega_j^2 = k_j(g+\sigma k_j^2)T_j$ with $T_j = \tanh k_j d$ ($j = 1,2$). A_j ($j = 1,2$) are yet to be determined amplitude functions corresponding, respectively, to the chosen (0,1) and (0,2) modes. The condition for the SHG requires $\omega_2 = 2\omega_1$, which corresponds to (2.10), discussed in § 2.1.

At the second order ($j = 1$), the solvability condition demands that the undetermined parameter v_g vanishes, as the modes we have chosen in (2.34) and (2.35) are two cutoff modes corresponding to the dispersion branches $\omega(k_x, \pi/b)$ and $\omega(k_x, 2\pi/b)$, respectively. The solution at the $j = 1$ order takes the same form as that at the leading order ($j = 0$) except that the amplitude functions A_j are changed into B_j, where B_j ($j = 1,2$) are two yet to be determined new amplitude functions of ξ and τ.

According to the solutions obtained above, at the next order ($j = 2$) we can calculate $P^{(2)}$, $Q^{(2)}$ and $R^{(2)}$. Then solving (2.15)–(2.17) with $j = 2$, we obtain

$$\begin{aligned}\phi^{(2)} = {} & \frac{\cosh k_1(z+d)}{\cosh k_1 d} \cos k_1 y \{C_1(\xi,\tau) \exp[-\mathrm{i}\omega_1 t)] + \text{c.c.}\} \\ & + \frac{\cosh k_2(z+d)}{\cosh k_2 d} \cos k_2 y \{C_2(\xi,\tau) \exp[-\mathrm{i}\omega_2 t)] + \text{c.c.}\} \\ & - \frac{(z+d)\sinh k_1(z+d)}{2k_1 \cosh k_1 d} \cos k_1 y (A_{1\xi\xi} \exp(-\mathrm{i}\omega_1 t) + \text{c.c.}) \\ & - \frac{(z+d)\sinh k_2(z+d)}{2k_2 \cosh k_2 d} \cos k_2 y (A_{1\xi\xi} \exp(-\mathrm{i}\omega_2 t) + \text{c.c.}),\end{aligned} \tag{2.36}$$

where C_1 and C_2 are two new amplitude functions yet to be determined. Substituting (2.36) into the boundary condition (2.29) with $j = 2$, we obtain two solvability conditions which are just the coupled amplitude equations for A_1 and A_2:

$$\mathrm{i}\frac{\partial A_1}{\partial \tau} + \tfrac{1}{2}\Gamma_1 \frac{\partial^2 A_1}{\partial \xi^2} + \mathrm{i}\Delta_1 A_1^* A_2 = 0, \tag{2.37}$$

$$\mathrm{i}\frac{\partial A_2}{\partial \tau} + \tfrac{1}{2}\Gamma_2 \frac{\partial^2 A_2}{\partial \xi^2} - \mathrm{i}\Delta_2 A_1^2 = 0, \tag{2.38}$$

with

$$\Gamma_1 = \frac{1}{\omega_1}\left\{ \frac{\omega_1^2}{2k_1^2 T_1}[T_1 + k_1 d(1 - T_1^2)] + \sigma k_1 T_1 \right\}, \tag{2.39}$$

$$\Gamma_2 = \frac{1}{\omega_2}\left\{ \frac{\omega_2^2}{2k_2^2 T_2}[T_2 + k_2 d(1 - T_2^2)] + \sigma k_2 T_2 \right\}, \tag{2.40}$$

$$\Delta_1 = \tfrac{1}{2}k_1^2 \left[1 - \tfrac{3}{2}T_1 T_2 + \frac{2T_1 + T_2}{2T_1} \right], \tag{2.41}$$

$$\Delta_2 = \tfrac{1}{4}k_2^2 \left[\tfrac{1}{4}(1 - 3T_1^2) + \frac{T_1}{T_2} \right]. \tag{2.42}$$

Note that Γ_j and Δ_j $(j = 1, 2)$ are positive real numbers. Using the transformation $u_j = \epsilon A_j$ $(j = 1, 2)$ with $\xi = \epsilon^{1/2}x$ and $\tau = \epsilon t$, (2.37) and (2.38) take the form

$$\mathrm{i}\frac{\partial u_1}{\partial t} + \tfrac{1}{2}\Gamma_1 \frac{\partial^2 u_1}{\partial x^2} + \mathrm{i}\Delta_1 u_1^* u_2 = 0, \tag{2.43}$$

$$\mathrm{i}\frac{\partial u_2}{\partial t} + \tfrac{1}{2}\Gamma_2 \frac{\partial^2 u_2}{\partial x^2} - \mathrm{i}\Delta_2 u_1^2 = 0. \tag{2.44}$$

If we allow a small frequency mismatch, i.e. $\omega_2 = 2\omega_1 + \delta\omega$ with $\delta\omega$ a small quantity of order ϵ, then the amplitude equations (2.43) and (2.44) are replaced by

$$\mathrm{i}\left(\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial x} \right) + \tfrac{1}{2}\Gamma_1 \frac{\partial^2 u_1}{\partial x^2} + \mathrm{i}\Delta_1 u_1^* u_2 \exp(\mathrm{i}\delta\omega t) = 0, \tag{2.45}$$

$$\mathrm{i}\left(\frac{\partial u_2}{\partial t} + v_2 \frac{\partial u_2}{\partial x} \right) + \tfrac{1}{2}\Gamma_2 \frac{\partial^2 u_2}{\partial x^2} - \mathrm{i}\Delta_2 u_1^2 \exp(-\mathrm{i}\delta\omega t) = 0, \tag{2.46}$$

where v_j $(j = 1, 2)$ are, respectively, the group velocities of the fundamental and the second-harmonic waves near $k_{x1} = 0$, $n = 1$ and $k_{x2} = 0$, $n = 2$. Equations (2.45) and (2.46) are coupled amplitude equations for the fundamental and the second-harmonic waves, respectively. Similar equations have been obtained by Karamzin & Sukhorukov (1974) in nonlinear optics and, recently, by Konotop & Malomed (2000) in nonlinear lattice dynamics. A significant difference between (2.45) and (2.46) and the amplitude equations obtained by McGoldrick (1970*b*) and Nayfeh (1970), for the usual SHG, is that in the present case we have included dispersion. We shall see in the following that the inclusion of dispersion results in quite different dynamic behaviour for the evolution of nonlinear water waves. In addition, from (2.39)–(2.42) we can see that our amplitude equations (2.45) and (2.46) are also valid for infinitely deep water (i.e. for $d \to \infty$ and hence we have $T_j \to 1$, $j = 1, 2$).

3. Hydrodynamic simulton solutions

In this section, we provide some exact solutions of the amplitude equations (2.45) and (2.46). We look for the solution with the following form:

$$u_1(x,t) = U_1(\eta)\exp(\mathrm{i}\theta_1), \tag{3.1}$$

$$u_2(x,t) = U_2(\eta)\exp(\mathrm{i}\theta_2), \tag{3.2}$$

where $\eta = Kx - \Omega t$, $\theta_j = K_j x - \Omega_j t + \phi_j$ $(j = 1,2)$ with $K_2 = 2K_1$, $\Omega_2 = 2\Omega_1 + \delta\omega$, $\phi_2 = 2\phi_1 - \pi/2$. K, Ω, K_1, Ω_1 and ϕ_1 are constants yet to be determined. Then (2.45) and (2.46) are transformed into

$$U_{1\eta\eta} + \alpha_1 U_1 U_2 - \beta_1 U_1 = 0, \tag{3.3}$$

$$U_{2\eta\eta} + \alpha_2 U_1^2 - \beta_2 U_2 = 0, \tag{3.4}$$

with

$$\alpha_j = \frac{2\Delta_j}{\Gamma_j K^2}, \tag{3.5}$$

$$\beta_j = \frac{\Gamma_j K_j^2 - 2(\Omega_j - v_j K_j)}{\Gamma_j K^2} \tag{3.6}$$

$(j = 1,2)$ with $K_1 = (v_2 - v_1)/(\Gamma_1 - 2\Gamma_2)$ and $\Omega = v_1 K + \Gamma_1 K K_1$. Note that α_1 and α_2 are positive constants. A coupled soliton–soliton (i.e. simultaneous solitary waves or solitons for two wave components) solution of (3.3) and (3.4) reads

$$U_1 = \frac{6s_1}{\sqrt{\alpha_1\alpha_2}}\operatorname{sech}^2\eta, \tag{3.7}$$

$$U_2 = \frac{6}{\alpha_1}\operatorname{sech}^2\eta, \tag{3.8}$$

with $s_1 = \pm 1$. For the solution (3.7) and (3.8) to exist, the condition $\beta_1 = \beta_2 = 4$ is required, resulting in

$$K^2 = \frac{2(v_2 - v_1)K_1 + (2\Gamma_2 - \Gamma_1)K_1^2 - \delta\omega}{2(\Gamma_2 - 2\Gamma_1)}. \tag{3.9}$$

In this case the free-surface displacement takes the form

$$\zeta(x,y,t) = -\frac{12s_1}{\sqrt{\alpha_1\alpha_2}}\frac{k_1 T_1}{\omega_1}\operatorname{sech}^2(Kx - \Omega t)\cos k_1 y \sin[K_1 x - (\omega_1 + \Omega_1)t + \phi_1]$$
$$+\frac{12}{\alpha_1}\frac{k_2 T_2}{\omega_2}\operatorname{sech}^2(Kx - \Omega t)\cos k_2 y \cos[K_2 x - (\omega_2 + \Omega_2)t + 2\phi_1], \tag{3.10}$$

where ϕ_1 is a phase constant depending on the initial condition. The first (respectively, second) term of (3.10) on the right-hand side corresponds to the fundamental (respectively, second-harmonic) wave component. We see that each component of the excitation is a standing wave in the y-direction and a *bright* (above level) envelope soliton in the x-direction. We call the solution (3.10) the *hydrodynamic simulton*. If k_{x1} (k_{x2}) is exactly zero but $\delta\omega \neq 0$, then $v_1 = v_2 = 0$. Consequently, $K_1 = K_2 = 0$, $K^2 = \delta\omega/(4\Gamma_1 - 2\Gamma_2)$, $\Omega = 0$, $\Omega_1 = -2\Gamma_1 K^2$ and $\Omega_2 = -2\Gamma_2 K^2$. Then (3.10) represents a *hydrodynamic non-propagating simulton*, in which the oscillating frequencies of both the fundamental and second-harmonic waves are smaller than the lower cutoff frequencies of the corresponding linear modes. Shown in figure 3 is the surface displacement pattern corresponding to the hydrodynamic non-propagating simulton (3.10) when $d = 2\,\mathrm{cm}$, $s_1 = +1$, $\phi_1 = 0$ and $\delta\omega = 0.8\,\mathrm{Hz}$ at different times, $t_0 = 0$ (figure 3*a*), $t_1 = \pi/[4(\omega_1 + \Omega_1)]$ (figure 3*b*), and $t_2 = \pi/[2(\omega_1 + \Omega_1)]$ (figure 3*c*). The vertical coordinate in the figures is chosen as the dimensionless surface displacement $\zeta(x, y, t)/[\alpha_1\omega_2/(12k_2\Gamma_2)]$. Note that the value of the width b of the channel has been chosen as 1.2083 cm, determined by the SHG resonance condition (2.10). The interaction between the (0,1) and the (0,2) modes is clearly shown. Obviously, the non-propagating simulton can be taken as a two-mode breather of the system. Each point on the breather has a periodic oscillation with the oscillating frequency ω_1. Figure 4 shows such an oscillation of the points on $x = 0$. If $\delta\omega \neq 0$ and hence $\Omega \neq 0$, the simulton is travelling along the x-direction with the velocity Ω/K.

Equations (3.3) and (3.4) admit another type of simulton solution, e.g.

$$U_1 = -\frac{6s_1}{\sqrt{\alpha_1\alpha_2}}\left(\tfrac{2}{3} - \mathrm{sech}^2\eta\right), \tag{3.11}$$

$$U_2 = -\frac{6}{\alpha_1}\left(\tfrac{2}{3} - \mathrm{sech}^2\eta\right). \tag{3.12}$$

When the condition $\beta_1 = \beta_2 = -4$ is imposed, the parameter K must take the value

$$K^2 = \frac{2(v_2 - v_1)K_1 + (2\Gamma_2 - \Gamma_1)K_1^2 - \delta\omega}{2(2\Gamma_1 - \Gamma_2)}. \tag{3.13}$$

The free-surface displacement is now given by

$$\begin{aligned}\zeta = {} & \frac{12s_1}{\sqrt{\alpha_1\alpha_2}}\frac{k_1T_1}{\omega_1}\left[\tfrac{2}{3} - \mathrm{sech}^2(Kx - \Omega t)\right]\cos k_1y\,\sin[K_1x - (\omega_1 + \Omega_1)t + \phi_1] \\ & - \frac{12}{\alpha_1}\frac{k_2T_2}{\omega_2}\left[\tfrac{2}{3} - \mathrm{sech}^2(Kx - \Omega t)\right]\cos k_2y\,\sin[K_2x - (\omega_2 + \Omega_2)t + 2\phi_1].\end{aligned} \tag{3.14}$$

Hence in this case both the fundamental wave and the second-harmonic wave are *dark* (below level) envelope solitons. In particular, when $v_1 = v_2 = 0$, we have $K_1 = K_2 = 0$, $\Omega = 0$, $K^2 = -\delta\omega/(4\Gamma_1 - 2\Gamma_2)$, $\Omega_1 = 2\Gamma_1K^2$ and $\Omega_2 = 2\Gamma_2K^2$. Hence the solution (3.14) represents a non-propagating simulton with the oscillation frequency of the fundamental and the second-harmonic waves both larger than the lower cutoff frequency of the corresponding linear mode. Note that to make K^2 positive one should chose the frequency mismatch $\delta\omega < 0$ in this particular circumstance.

Formally, (3.3) and (3.4) also admit the following coupled soliton solution:

$$U_1 = \mathrm{i}\frac{6s_1}{\sqrt{-\alpha_1\alpha_2}}\mathrm{sech}\,\eta\tanh\eta, \tag{3.15}$$

$$U_2 = -\frac{6}{\alpha_1}\mathrm{sech}^2\eta. \tag{3.16}$$

However, such a solution cannot be physically realized because in our system both α_1 and α_2 are positive.

Physically, the formation of the hydrodynamic simultons provided in (3.10) and (3.14) results in an effective energy transfer between the fundamental and the second-harmonic wave components. The energy is then transferred back again to each wave mode itself as a consequence of dispersion, and hence we have a cascading process or mutual self-trapping.

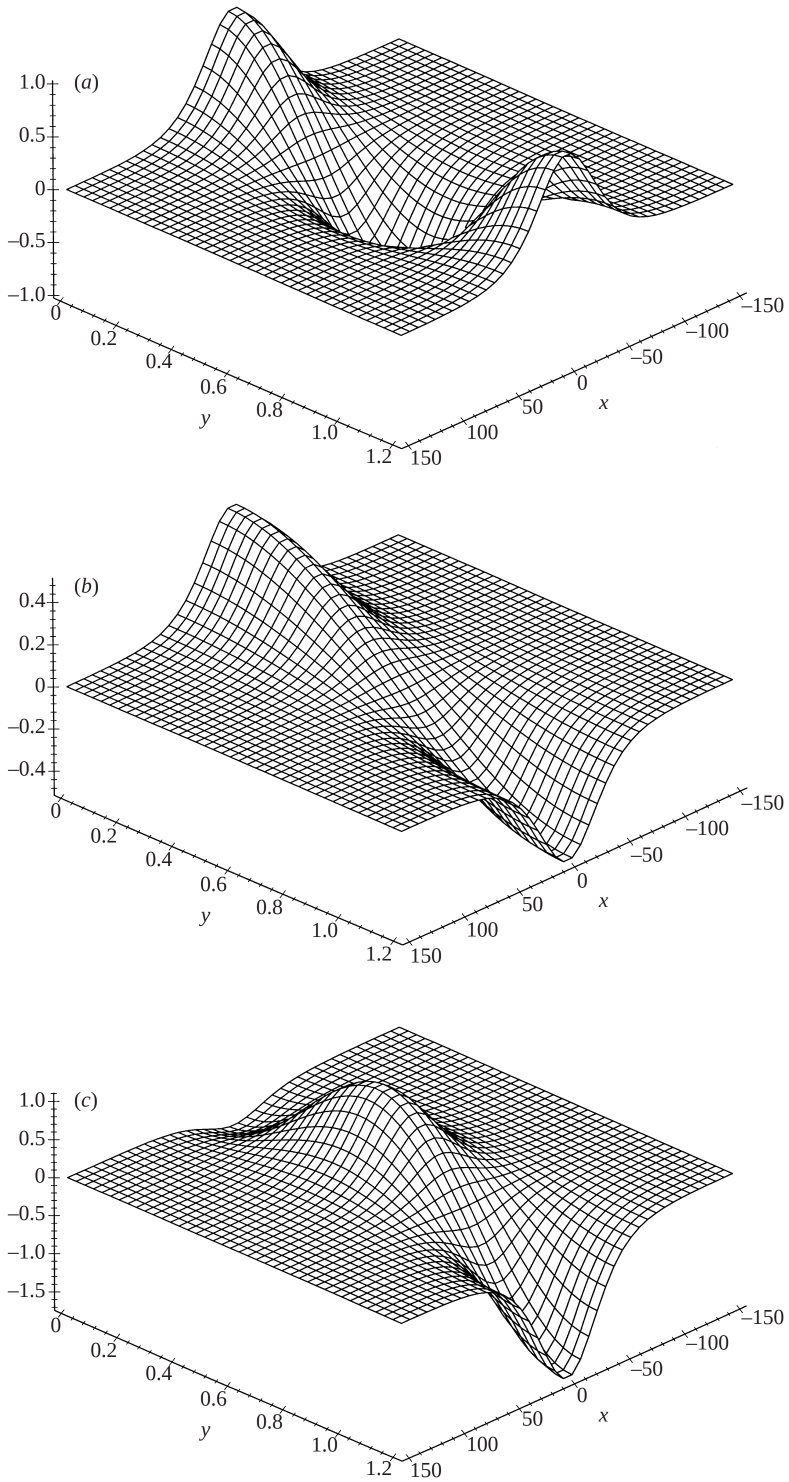

FIGURE 3. The dimensionless surface displacement $[(\alpha_1\omega_2)/(12k_2T_2)]\zeta(x,y,t)$ corresponding to the non-propagating simulton for $d = 2\,\mathrm{cm}$, $s_1 = +1$, $\phi_1 = 0$ and $\delta\omega = 0.8\,\mathrm{Hz}$ at times (*a*) $t = 0$, (*b*) $t = \pi/[4(\omega_1 + \Omega_1)]$, (*c*) $t = \pi/[2(\omega_1 + \Omega_1)]$.

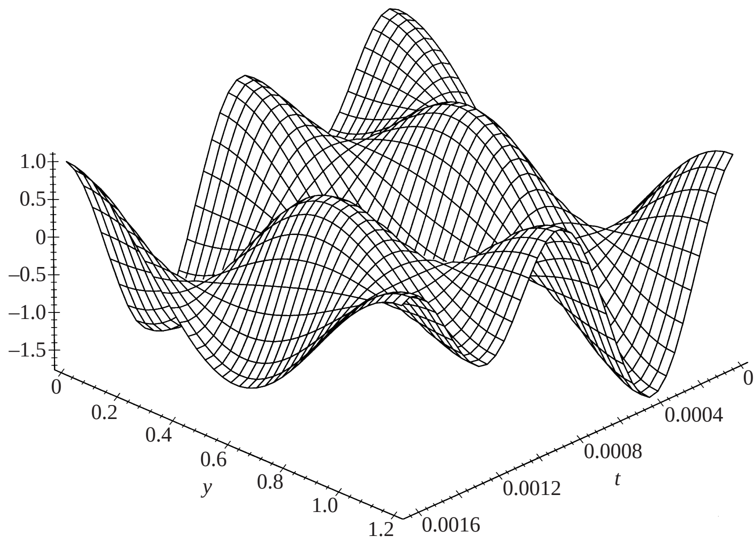

FIGURE 4. Periodic oscillation of the points on $x = 0$ in the water surface corresponding to the hydrodynamic simulton. The parameters are the same as those in figure 3.

4. Parametrically excited hydrodynamic simultons

Wu, Keolian & Rudnick (1984) made the important experimental finding of a soliton in a parametrically excited fluid layer. Their system was a narrow and long rectangular channel filled with water to finite depth. When the system is driven vertically with a low-frequency loudspeaker, or a similar low-frequency vibrating support, a non-propagating soliton appears on the free surface of the water, which is a nonlinearly modulated standing wave related to the (0,1)-mode of the system. Since our system, described in § 2.1, parallels that used by Wu *et al.* (1984), we expect that a parametrically excited hydrodynamic simulton could also be observed following our findings described above.

As in Wu *et al.* (1984) we assume that a long water channel is placed on a vibrating support. The width b of the channel is designed such that the phase-matching condition of the SHG (2.10) is satisfied. As mentioned in § 2.1, for water at room temperature the phase-matching condition for the SHG requires $b = 1.21$ cm if the static water depth is $d = 2$ cm. To excite hydrodynamic simultons we apply in the vertical direction an external double-frequency drive $z_0(t) = a_{e1}\cos(2\omega_e t) + a_{e2}\cos(4\omega_e t)$, with ω_e near ω_1, the frequency of the fundamental wave. The equations of motion of the system are still (2.1)–(2.5) but the gravitational acceleration, g, is now replaced by a time-dependent parameter, $g + \ddot{z}_0(t)$. Using a similar asymptotic expansion as in (2.11)–(2.14) and assuming

$$4\omega_e^2 a_{e1}/g = \gamma_1\epsilon, \quad 16\omega_e^2 a_{e2}/g = \gamma_2\epsilon, \tag{4.1}$$

and

$$(\omega_e - \omega_1)/\omega_1 = \gamma_3\epsilon \tag{4.2}$$

with $\gamma_j (j = 1, 2, 3)$ of order unity, we obtain the following coupled amplitude equations:

$$\mathrm{i}\frac{\partial A_1}{\partial\tau} + \tfrac{1}{2}\Gamma_1\frac{\partial^2 A_1}{\partial\xi^2} + \mathrm{i}\Delta_1 A_1^* A_2 - \tilde{\nu}_1 A_1^* \exp(-2\mathrm{i}\omega_1\gamma_3\tau) = 0, \tag{4.3}$$

$$\mathrm{i}\frac{\partial A_2}{\partial\tau} + \tfrac{1}{2}\Gamma_2\frac{\partial^2 A_2}{\partial\xi^2} - \mathrm{i}\Delta_2 A_1^2 - \tilde{\nu}_2 A_2^* \exp(-4\mathrm{i}\omega_2\gamma_3\tau) = 0, \tag{4.4}$$

where $\tilde{v}_j = g\gamma_j k_j T_j/(4\omega_j)$ $(j = 1, 2)$. Γ_j and Δ_j $(j = 1, 2)$ are the same as those given in (2.39)–(2.42). A_1 and A_2 are still the envelope functions of the fundamental and second-harmonic wave components, respectively. If we allow a small frequency mismatch in the SHG resonance condition, (4.3) and (4.4) should become

$$\mathrm{i}\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x} + d_1\right)u_1 + \tfrac{1}{2}\Gamma_1\frac{\partial^2 u_1}{\partial x^2} + \mathrm{i}\Delta_1 u_1^* u_2 \exp(\mathrm{i}\delta\omega t) - v_1 u_1^* \exp(-2\mathrm{i}\delta\omega_e t) = 0, \quad (4.5)$$

$$\mathrm{i}\left(\frac{\partial}{\partial t} + v_2\frac{\partial}{\partial x} + d_2\right)u_2 + \tfrac{1}{2}\Gamma_2\frac{\partial^2 u_2}{\partial x^2} - \mathrm{i}\Delta_2 u_1^2 \exp(-\mathrm{i}\delta\omega t) - v_2 u_2^* \exp(-4\mathrm{i}\delta\omega_e t) = 0 \quad (4.6)$$

when returning to the original variables, where $(u_1, u_2) = \epsilon(A_1, A_2)$, $v_1 = \omega_e a_{e1} k_1 T_1$, $v_2 = 4\omega_e a_{e2} k_2 T_2$ and $\delta\omega_e = \omega_e - \omega_1$. As in (2.45) and (2.46), v_1 and v_2 are the group velocities of the fundamental and the second-harmonic waves near $k_{x1} = 0$, $k_{y1} = \pi/b$ and $k_{x2} = 0$, $k_{y2} = 2\pi/b$, respectively. Since damping always exists in the system, for simplicity, following Miles (1984) and Huang *et al.* (1990), we have phenomenologically included two *ad hoc* damping parameters d_1 and d_2.

To solve the parametrically driven amplitude equations (4.5) and (4.6) we assume

$$u_1 = U_1(\eta)\exp[\mathrm{i}\theta_1 - \mathrm{i}(\omega_e - \omega_1)t], \quad (4.7)$$

$$u_2 = U_2(\eta)\exp[\mathrm{i}\theta_2 - 2\mathrm{i}(\omega_e - \omega_1)t], \quad (4.8)$$

with $\eta = Kx - \Omega t$ and $\theta_j = K_j x - \Omega_j t + \phi_j$ $(j = 1, 2)$. Substituting (4.7) and (4.8) into (4.5) and (4.6) and then comparing the real and imaginary parts gives $\Omega = \Omega_1 = \Omega_2 = 0$, $K_1 = K_2 = 0$, $v_1 = v_2 = 0$, $\omega_2 = 2\omega_1$ and $\phi_2 = 2\phi_1 - \pi/2$. Hence only non-propagating solutions are possible. Under these conditions, (4.5) and (4.6) become (3.3) and (3.4) with the same α_j $(j = 1, 2)$ given in (3.5) but with

$$\beta_1 = 2\frac{v_1\cos 2\phi_1 - (\omega_e - \omega_1)}{\Gamma_1 K^2}, \quad (4.9)$$

$$\beta_2 = 2\frac{v_2\cos 2\phi_2 - 2(\omega_e - \omega_1)}{\Gamma_2 K^2}. \quad (4.10)$$

The condition $\phi_2 = 2\phi_1 - \pi/2$ corresponds to

$$\left(\frac{d_2}{v_2}\right)^2 = 4\left(\frac{d_1}{v_1}\right)^2\left[1 - \left(\frac{d_1}{v_1}\right)^2\right]. \quad (4.11)$$

It gives a constraint for the driving amplitudes, a_{e1} and a_{e2}, and the damping parameters, d_1 and d_2, and, accordingly, must be suitably adjusted in experiment.

We obtain two types of non-propagating simulton solutions. One of them is $U_1 = (6s_1/\sqrt{\alpha_1\alpha_2})\mathrm{sech}^2(Kx)$ and $U_2 = -(6/\alpha_1)\mathrm{sech}^2(Kx)$ with $K^2 = [\mu_1\cos 2\phi_1 - (\omega_e - \omega_1)]/(2\Gamma_1)$. For this soliton–soliton solution to be valid the driving frequency and the driving amplitude must satisfy the relation $\omega_e - \omega_1 = (\Gamma_2 v_1\cos 2\phi_1 - \Gamma_1 v_2 \cos 2\phi_2)/(\Gamma_2 - 2\Gamma_1)$, with $\sin 2\phi_j = -\eta_j/v_j$ $(j = 1, 2)$. The free-surface displacement in this case takes the form

$$\zeta(x, y, t) = \frac{12s_1}{\sqrt{\alpha_1\alpha_2}}\frac{k_1 T_1}{\omega_1}\mathrm{sech}^2(Kx)\cos k_1 y \sin(\omega_e t - \phi_1)$$
$$-\frac{12}{\alpha_1}\frac{k_2 T_2}{\omega_2}\mathrm{sech}^2(Kx)\cos k_2 y \cos(2\omega_e t - 2\phi_1), \quad (4.12)$$

with $\phi_1 = -(1/2)\sin_{-1}(\eta_1/v_1)$.

The other type of non-propagating simulton is $U_1 = -(6s_1/\sqrt{\alpha_1\alpha_2})[2/3-\mathrm{sech}^2(Kx)]$ and $U_2 = -(6/\alpha_1)[2/3 - \mathrm{sech}^2(Kx)]$ with $K^2 = (\omega_e - \omega_1 - v_1\cos 2\phi_1)/(2\Gamma_1)$ and $\omega_e - \omega_1 = (\Gamma_2 v_1 \cos 2\phi_1 - \Gamma_1 v_2 \cos 2\phi_2)/(2\Gamma_1 - \Gamma_2)$. In this case the surface displacement reads

$$\zeta(x, y, t) = -\frac{12s_1}{\sqrt{\alpha_1\alpha_2}}\frac{k_1 T_1}{\omega_1}[\tfrac{2}{3} - \mathrm{sech}^2(Kx)]\cos k_1 y \sin(\omega_e t - \phi_1)$$
$$-\frac{12}{\alpha_1}\frac{k_2 T_2}{\omega_2}[\tfrac{2}{3} - \mathrm{sech}^2(Kx)]\cos k_2 y \cos(2\omega_e t - 2\phi_1). \quad (4.13)$$

The above results show that, using parametric driving with appropriate conditions imposed on the corresponding driving amplitude and driving frequency, it is indeed possible to observe hydrodynamic simultons in a water channel.

5. Discussion and summary

We have investigated the second-harmonic generation (SHG) of nonlinear surface water waves in a long rectangular channel filled with water to finite depth. Generalizing earlier work by Simmons (1969), McGoldrick (1970*a*, *b*), Nayfeh (1970) and Miles (1984) we have shown that a new type of nonlinear excitation, simultaneous capillary–gravity (simultons or quadratic solitons) waves, can be excited. Taking into account wave dispersion, two coupled nonlinear equations for the amplitudes of the fundamental and second-harmonic waves, (2.45) and (2.46), have been derived and two types of hydrodynamic simulton solutions have been explicitly provided. We have also studied the possibility of parametrically exciting hydrodynamic simultons in a water channel. Incorporating damping and having the liquid channel parametrically excited by vertical vibration with suitable frequency and amplitude, the corresponding evolution equations, (4.5) and (4.6), have been derived, thus generalizing (2.45) and (2.46). Parametrically excited simulton solutions have been obtained and an experiment for observing such new nonlinear hydrodynamic excitations has been suggested as a straightforward extension of earlier experimental work by Wu *et al.* (1984).

Unlike the conventional solitons in deep water, in which the generation of a *single* soliton is due to the self-trapping of linear plane waves, the mechanism for the formation of the hydrodynamic simultons presented here is through cascading between two wave modes. In this process, the fundamental and the second-harmonic waves (with frequency ω_1 and ω_2, respectively) interact with each other through repeated three-wave interactions. For example, the energy of the fundamental wave is first up-converted to the second-harmonic wave (through the sum-frequency process $\omega + \omega = 2\omega$) and then down-converted (through the difference-frequency process $2\omega - \omega = \omega$), resulting in *mutual* self-trapping of the *two* waves thus leading to the simultaneous appearance of two hydrodynamic solitons (hence the term simultons). Besides the usual condition for an SHG, i.e. the phase matching given in (2.7) and (2.8), the occurrence of hydrodynamic simultons requires an additional group-velocity matching condition (2.9). We have also shown that wave dispersion plays a significant role in the generation of this type of excitation.

Gottwald, Grimshaw & Malomed (1997, 1998) have considered parametric envelope solitons in systems of coupled Korteweg–de Vries and Kadomtsev–Petviashvili-like equations, describing the coupling of internal waves in a shallow stably stratified fluid and thus valid only in the long-wave approximation. Their coupled envelope solitons are in fact the simultons in shallow water. Our systems described in § 2 and § 4 are long rectangular waveguides filled with liquid to finite depth in a channel open to air with dispersion, gravity and surface tension taken into account. The results presented in §§ 2–4 can be easily extended to an infinitely deep water layer.

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